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AN ANALOG OF THE BJERKNES THEOREM IN MAGNETOGASDYNAMICS

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In studying the influence of a magnetic force on the formation of velocity vortices in a magnetogasdynamic flow, one usually considers the vorticity equation (see [1], for example). Another approach involves study of velocity circulation over a closed contour. With this approach, consideration is usually restricted to verification of the necessary condition of fulfillment of the Thompson theorem, namely, the potentiality of the magnetic force $(\mathbf{j} \times \mathbf{H})/\rho$. In some cases, however, this approach also makes it possible to study in more detail the magnetic mechanism of vortex formation.

We integrate the equation of motion $d\mathbf{v}/dt = \mathbf{F}/\rho$ along an arbitrary closed fluid contour L, taking into account the fact that the time derivative of velocity circulation equals acceleration circulation along the same contour ($\mathbf{F}$ is the total force acting on a unit volume). As a result, we have

$$d\Gamma/dt = \gamma + \tilde{\gamma}, \quad \Gamma = \oint_L \mathbf{v}\, d\mathbf{l}, \quad \gamma = \oint_L (c\rho)^{-1} (\mathbf{j} \times \mathbf{H})\, d\mathbf{l}.$$

Here $\tilde{\gamma}$ is the contribution of other constituents of the force $\mathbf{F}$ to the change in circulation of Γ.

We consider two-dimensional flows in a transverse magnetic field, one in the plane (x, y) for $\mathbf{H} = H\mathbf{e}_z$ and another in the plane (r, z) for $\mathbf{H} = H\mathbf{e}_\varphi$. In both cases, $(c\rho)^{-1} (\mathbf{j} \times \mathbf{H}) = -æ\nabla I$, where $æ = H/(4\pi\, \rho\, r^\nu)$, $I = Hr^\nu$, and $\nu = 0$ for plane flow and $\nu = 1$ for axisymmetric flow. Thus,

$$\gamma = -\oint_L æ\, dI. \tag{1}$$

We consider a system of tubes formed in space by intersection of the surfaces $æ =$ const and $I =$ const. For flows with infinite electrical conductivity, $æ$ is the freezing-in integral, which is constant along the fluid-particle trajectory. For this reason, we call the surfaces $æ =$ const isomagnetic, following the terminology in [2]. We call the surfaces $I =$ const the current surfaces, inasmuch as they carry a current ($\mathbf{j}\nabla I = 0$). The above-mentioned tubes are called current-isomagnetic. Note that for plane flow of constant density, the current surfaces coincide with the isomagnetic ones and $\gamma \equiv 0$.

We calculate the integral $\oint æ dI$ along the contour ABCD, a section of the tube (see the Fig. 1):

$$-\oint_{\mathrm{ABCD}} æ dI = (æ_1 - æ_2)(I_1 - I_2) = \delta \Delta æ \Delta I. \tag{2}$$

Here $\Delta æ = |æ_1 - æ_2|$, $\Delta I = |I_1 - I_2|$; and $\delta = 1$, if the positive direction in tracing the boundary ABCD (counterclockwise) coincides with the direction of rotation from the vector ∇I to the vector $\nabla æ$ along the shortest path, and $\delta = -1$ if these directions are opposite. The isomagnetic current tube is called positive in the first case and negative in the second case. It is obvious that an arbitrary contour L can encompass both positive and negative tubes. Assuming (without loss of generality) that $\Delta æ =$ const and $\Delta I =$ const and taking into account (2), we recast (1) as $\gamma = (N_+ - N_-)\, \Delta æ \Delta I$, where N_+ and N_- are the numbers of positive and negative tubes, respectively. Thus, for the cases of MGD flows considered, the contribution of the magnetic force to the change in velocity circulation along a closed fluid contour with time is determined by the

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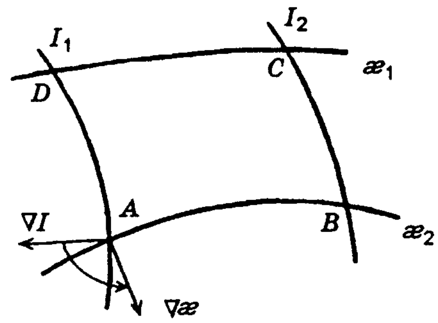

Fig. 1

difference between the numbers of positive and negative isomagnetic current tubes enclosed by the contour. This statement establishes a magnetic mechanism of vortex formation that involves noncoincidence between the isomagnetic and current surfaces and is an MGD analog of the Bjerknes theorem [3, 4]. In particular, in isomagnetic flow ($æ \equiv$ const), a magnetic force does not change the circulation. From this point of view, the isomagnetic character of the flows in question plays the same role as the incompressibility of gas-dynamic flow; the dependence $I = I(æ)$ is an analog of barotropy.

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